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Tensionless supersymmetric M2 branes in $\mathrm{AdS}_4 \times S^7$ and giant diabolo

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Abstract: We find various supersymmetric configurations of toroidal M2 brane solutions in $\mathrm{AdS}_4 \times S^7$ or, more generally, in $\mathrm{AdS}_4 \times S^7/\mathbb{Z}_k$. In this class we identify solutions preserving 1/4 and 1/8 supersymmetries of the background. The supersymmetric M2 branes have angular momenta and winding on S^7, and null world-volumes. In certain cases they collapse to string-like configurations. These configurations can be viewed as a higher-dimensional (membrane) analog of BMN states. We compute the energy and angular momenta, showing that all supersymmetric configurations obey the BPS relation $E = J/R$, $J \equiv \sum_{i=1}^{4} |J_i|$ with $E, J \to \infty$. Finally, we also study another class of supersymmetric M2-branes, including uncompact rotating membranes of "diabolo" shape.

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Contents

1 Introduction

An important aspect of the AdS/CFT correspondence [1–3] is understanding the precise map between supersymmetric states in the CFT and on the gravity side. In the last years, there was an impressive progress in this direction. In particular, the supergravity spectrum on $AdS_5 \times S^5$ [4] was put [3] in precise correspondence with the spectrum of 1/2 BPS operators of $N = 4$ super Yang-Mills.

The correspondence between CFT operators and string states on AdS was generalized to various sectors in [5] for (near BPS) collapsed string configurations and for more general extended string states in numerous works (for reviews, see, e.g. [6]). The correspondence between the spectra applies also to extended supersymmetric brane configurations, such as BPS D brane configurations or giant gravitons, and the identification of the corresponding operators led to important insights on the nature of the AdS/CFT correspondence [7–10].

The recent discovery of the ABJM superconformal field theory describing the physics of multiple membranes probing an orbifold space [11] provides an extremely interesting setup to understand properties of AdS/CFT correspondence and of M-theory from a new perspective. In a recent work [12] BPS M2 brane configurations representing giant tori were constructed. The corresponding states carry a large amount of angular momentum and D0 brane charge. The corresponding ABJM field theory interpretation was discussed in [13]. In this paper we will look for different types of supersymmetric configurations. The general M2-brane solutions discussed here have a structure which is analog to that of the (non-supersymmetric) circular strings of [14]. These type of M2 brane solutions were investigated in [15]. The configurations can also be viewed as (toroidal) giant gravitons. Here we will show that there is an important subclass of solutions which are supersymmetric (general aspects of supersymmetric giant gravitons are discussed in [16]). This subclass of solutions has the property of having a vanishing determinant for the induced metric, i.e. a null world-volume. This is possible only for a tensionless membrane. They may be viewed as the large J limit of regular membranes. The solutions are the precise membrane analog of the tensionless strings discussed in [17]. They also represent a higher dimensional version of the BMN states.

This paper is organized as follows. In section 2 we review the $AdS_4 \times S^7$ and $AdS_4 \times S^7/\mathbb{Z}_k$ backgrounds and their supersymmetries. In section 3.1 we discuss the classical equations of motion for an M2 brane moving in $AdS_4 \times S^7$. In section 3.2 we introduce our general ansatz that describes an M2 brane that rotates and winds in S^7, and in section 3.3 we explicitly find the values of winding number and angular velocities that solve all equations of motion. In section 4 we derive the BPS bound for the energy from the superalgebra. In section 5 we find the energy formula for our membrane solutions and show that in the supersymmetric limit they reduce to the expected BPS form derived in section 4. In section 6 we identify the subclass of solutions which preserve some fraction of supersymmetry. Section 6.2 describes a class of regular supersymmetric membrane solutions, while section 6.3 discusses collapsed membrane configurations. In section 7 the solutions are adapted to the case of $AdS_4 \times S^7/\mathbb{Z}_k$ and, by dimensional reduction, we obtain supersymmetric states in $AdS_4 \times CP^3$. In section 8 we revisit the giant torus rotating membrane solution

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found in [12] and show that in a certain region of the parameters the rotating membrane opens up taking a "diabolo" shape.[1] We exhibit the solution in cylindrical coordinates, where it has a simpler form, and present a convenient characterization of the torus, spiky membrane, diabolo, cylinder and hyperboloid regimes in terms of a single parameter (the last three solutions did not appear in [12]). In section 9 we present a summary of our results. Appendix A contains additional details of the calculations omitted in the main text, appendix B contains an alternative derivation of the supersymmetries of the collapsed membranes by treating them as effective strings and in appendix C we give the expressions for the charges of the solutions of section 8.

2 Properties of $\mathrm{AdS}_4 \times S^7$ and $\mathrm{AdS}_4 \times S^7/\mathbb{Z}_k$ backgrounds

The space $\mathrm{AdS}_4 \times S^7$ can be represented by the metric

$$\mathrm{d}s^2 = \frac{R^2}{4}\big(\mathrm{d}s^2_{\mathrm{AdS}_4} + 4\,\mathrm{d}\Omega_7^2\big)\,, \tag{2.1}$$

where $R = \ell_p (2^5\pi^2 N)^{1/6}$, $\mathrm{d}\Omega_7^2$ stands for the unit radius S^7 round metric, and

$$\mathrm{d}s^2_{\mathrm{AdS}_4} = -(1+r^2)\,\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{1+r^2} + r^2\big(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2\big)\ . \tag{2.2}$$

The 4-form flux reads

$$F^{(4)} = -\frac{3}{8}R^3\, r^2\,\sin\theta\,\mathrm{d}t\wedge\mathrm{d}r\wedge\mathrm{d}\theta\wedge\mathrm{d}\varphi\ . \tag{2.3}$$

We can parametrize the S^7 using four complex coordinates, Z^i, which satisfy

$$|Z^1|^2 + |Z^2|^2 + |Z^3|^2 + |Z^4|^2 = R^2\ . \tag{2.4}$$

Choosing

$$Z^i = R\,\mu_i\,\mathrm{e}^{i\xi^i}\,,\qquad \sum_{i=1}^{4}{\mu_i}^2 = 1\,, \tag{2.5}$$

the coordinates μ_i can be written in terms of hyper-spherical coordinates. A possible choice is

$$\begin{aligned} \mu_1 &= \sin\alpha\,,\\ \mu_2 &= \cos\alpha\sin\beta\,,\\ \mu_3 &= \cos\alpha\cos\beta\sin\gamma\,,\\ \mu_4 &= \cos\alpha\cos\beta\cos\gamma\ . \end{aligned} \tag{2.6}$$

In these coordinates, the full metric reads

$$\begin{aligned} \mathrm{d}s^2 = \frac{R^2}{4}&\left\{-(1+r^2)\,\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{1+r^2} + r^2\big(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2\big)\right\}\\ &+ R^2\left\{\mathrm{d}\alpha^2 + \cos^2\alpha\,\mathrm{d}\beta^2 + \cos^2\alpha\ \cos^2\beta\,\mathrm{d}\gamma^2 + \sum_{i=1}^{4}{\mu_i}^2\,\mathrm{d}\xi^{i2}\right\}\ . \end{aligned} \tag{2.7}$$

[1] The diabolo consists of a spool whirled and tossed on a string (it illustrates angular momentum conservation and it was said to be the favorite toy of Maxwell).

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M-theory on $\mathrm{AdS}_4 \times S^7/\mathbb{Z}_k$ is obtained by identification under the $\mathbb{Z}_k$ orbifold action

$$Z^i \to e^{i\frac{2\pi}{k}}\, Z^i \qquad \Longleftrightarrow \qquad \xi^i \to \xi^i + \frac{2\pi}{k}\,, \tag{2.8}$$

with integer k. The solution represents the gravity dual of N M2-branes probing a $\mathbf{C}^4/\mathbb{Z}_k$ singularity, with R equal to $\ell_p(2^5\pi^2 Nk)^{1/6}$. To connect with the ABJM theory it is useful to introduce CP^3 adapted variables. By completing squares we can write

$$\mathrm{d}\Omega_7^2 = \mathrm{d}s^2_{CP^3} + (\mathrm{d}y + A)^2\,, \tag{2.9}$$

where $\mathrm{d}A = 2\mathcal{J}$ and $\mathcal{J}$ is the Kähler form of CP^3. We introduce a new set of coordinates adapted to CP^3, defined by

$$\begin{aligned}
\varphi_1 &= \xi^1 - \xi^2\,, & \varphi_2 &= \xi^3 - \xi^4\,,\\
y &= \frac{1}{4}\big(\xi^1 + \xi^2 + \xi^3 + \xi^4\big)\,, & \psi &= \frac{1}{2}\big(\xi^1 + \xi^2 - \xi^3 - \xi^4\big)\,,\\
\mu_1 &= \cos\zeta\cos\frac{\theta_1}{2}\,, & \mu_2 &= \cos\zeta\sin\frac{\theta_1}{2}\,,\\
\mu_3 &= \sin\zeta\cos\frac{\theta_2}{2}\,, & \mu_4 &= \sin\zeta\sin\frac{\theta_2}{2}\,.
\end{aligned} \tag{2.10}$$

By reducing along y, we get type IIA strings on $\mathrm{AdS}_4 \times CP^3$,

$$\mathrm{d}s^2 = \tilde{R}^2\big(\mathrm{d}s^2_{\mathrm{AdS}_4} + 4\,\mathrm{d}s^2_{CP^3}\big)\,, \qquad \tilde{R}^2 = \frac{1}{4k}R^3\,, \tag{2.11a}$$

$$\begin{aligned}
\mathrm{d}s^2_{CP^3} &= \mathrm{d}\zeta^2 + \cos^2\zeta\,\sin^2\zeta\,\left(\mathrm{d}\psi + \frac{1}{2}\cos\theta_1\,\mathrm{d}\varphi_1 - \frac{1}{2}\cos\theta_2\,\mathrm{d}\varphi_2\right)^2\\
&\quad + \frac{1}{4}\cos^2\zeta\,\left(\mathrm{d}\theta_1^2 + \sin^2\theta_1\,\mathrm{d}\varphi_1^2\right) + \frac{1}{4}\cos^2\zeta\,\left(\mathrm{d}\theta_2^2 + \sin^2\theta_2\,\mathrm{d}\varphi_2^2\right)\,,
\end{aligned} \tag{2.11b}$$

with a one- and 3-form RR potentials and dilaton given by [18]

$$C^{(1)} = \frac{k}{2}\,\Big[(\cos^2\zeta - \sin^2\zeta)\,\mathrm{d}\psi + \cos^2\zeta\,\cos\theta_1\,\mathrm{d}\varphi_1 + \sin^2\zeta\,\cos\theta_2\,\mathrm{d}\varphi_2\Big] = k\,A\,, \tag{2.12}$$

$$C^{(3)} = \frac{k}{2}\,\tilde{R}^2\,r^3\sin\theta\,\mathrm{d}t\wedge\mathrm{d}\theta\wedge\mathrm{d}\varphi\,, \tag{2.13}$$

$$\mathrm{e}^{2\phi} = \frac{4\,\tilde{R}^2}{k^2}\,. \tag{2.14}$$

We now describe the supersymmetries of the background. Our conventions for the Clifford algebras is such that $\{\Gamma_\mu,\,\Gamma_\nu\} = 2\,g_{\mu\nu}$, where $g_{\mu\nu}$ is given by (2.7), and $\{\gamma_\mu,\,\gamma_\nu\} = 2\,\eta_{\mu\nu}$ is the standard flat space-time Dirac algebra. We also define $\hat{\gamma} = -\gamma_{0123}$. This allows us to write

$$\begin{aligned}
\Gamma_t &= \frac{R}{2}\sqrt{1+r^2}\,\gamma_0\,, & \Gamma_r &= \frac{R}{2}\frac{1}{\sqrt{1+r^2}}\,\gamma_1\,,\\
\Gamma_\theta &= \frac{R}{2}\,r\,\gamma_2\,, & \Gamma_\varphi &= \frac{R}{2}\,r\sin\theta\,\gamma_3\,,\\
\Gamma_\alpha &= R\,\gamma_4\,, & \Gamma_\beta &= R\cos\alpha\,\gamma_5\,,\\
\Gamma_\gamma &= R\cos\alpha\cos\beta\,\gamma_6\,, & \Gamma_{\xi^i} &= R\,\mu_i\,\gamma_{i+6}\,.
\end{aligned} \tag{2.15}$$

The Killing spinors of this background are given by

$$\epsilon = \mathcal{M}\,\epsilon_0\,,$$
$$\mathcal{M} \equiv M_\alpha M_\beta M_\gamma \left(\prod_{i=1}^{4} M_i\right) M_r M_t M_\theta M_\varphi\ . \tag{2.16}$$

Here ϵ_0 is an arbitrary constant Majorana spinor, and the M_μ's are the exponentiation of generators of translations in the μ-direction,

$$\begin{aligned} M_t &= \mathrm{e}^{\frac{t}{2}\hat\gamma\gamma_0}\,, \quad M_r = \mathrm{e}^{\frac{\bar r}{2}\hat\gamma\gamma_1}\,, \quad M_\theta = \mathrm{e}^{\frac{\theta}{2}\gamma_{12}}\,, \quad M_\varphi = \mathrm{e}^{\frac{\varphi}{2}\gamma_{23}}\,, \\ M_\alpha &= \mathrm{e}^{\frac{\alpha}{2}\hat\gamma\gamma_4}\,, \quad M_\beta = \mathrm{e}^{\frac{\beta}{2}\hat\gamma\gamma_5}\,, \quad M_\gamma = \mathrm{e}^{\frac{\gamma}{2}\hat\gamma\gamma_6}\,, \quad M_i = \mathrm{e}^{\frac{\xi^i}{2}\mathbb{X}_i}\ . \end{aligned} \tag{2.17}$$

where we have defined $r = \sinh\hat r$, and introduced[2]

$$(\mathbb{X}_i) \equiv (\gamma_{47},\ \gamma_{58},\ \gamma_{69},\ \hat\gamma\gamma_{10})\,, \qquad \mathbb{X}_1\,\mathbb{X}_2\,\mathbb{X}_3\,\mathbb{X}_4 = -1. \tag{2.18}$$

Next, consider the $\mathbb{Z}_k$ orbifold action (2.8), which only affects to the ξ^i angular variables. Let us define the eigenvalues of $\mathbb{X}_1$, $\mathbb{X}_2$ and $\mathbb{X}_3$ to be $i\varsigma_i$. Since $\mathbb{X}_i^2 = -1$, it must be $\varsigma_i = \pm 1$. The spinors in (2.16) with $\varsigma_1 = \varsigma_2 = \varsigma_3$ are projected out by the projection (2.8) with $k > 2$, henceforth 24 Killing spinors (3/4 of the original 32) survive the orbifold action.

3 A class of M2 brane configurations

3.1 Action and equations of motion

Let Y^μ, with $\mu = 0, \cdots, 4$, be the embedding coordinates in the AdS piece of the space, and X^k, $k = 1, \cdots, 8$, the ones corresponding to the 7-sphere. The membrane action reads [15]

$$\begin{aligned} S =& \frac{T_2}{2}\int \mathrm{d}^3\sigma\,\Big(-\sqrt{-h}\,h^{\alpha\beta}\big(\eta_{\mu\nu}\,\partial_\alpha Y^\mu\partial_\beta Y^\nu + \delta_{kj}\,\partial_\alpha X^k\partial_\beta X^j\big) + \sqrt{-h} \\ &+ \tilde\Lambda\,\Big(\eta_{\mu\nu}\,Y^\mu Y^\nu + \frac{R^2}{4}\Big) + \Lambda\,(X^kX^k - R^2)\Big) + T_2\int C^{(3)}|_{pullback}\ . \end{aligned} \tag{3.1}$$

We choose $\eta_{\mu\nu} = \mathrm{diag}(-1,1,1,1,-1)$. $\tilde\Lambda$ and Λ are Lagrange multipliers that enforce the conditions

$$\eta_{\mu\nu}\,Y^\mu Y^\nu = -\frac{R^2}{4}\,, \qquad \sum_{k=1}^{4}(X^k)^2 = R^2\,, \tag{3.2}$$

respectively, thus defining the $AdS_4 \times S^7$ space.

Using the formula $\delta h = -h h_{\alpha\beta}\delta h^{\alpha\beta}$ one finds that the equation of motion for the world-volume metric gives

$$h_{\alpha\beta} = \eta_{\mu\nu}\,\partial_\alpha Y^\mu\partial_\beta Y^\nu + \delta_{kj}\,\partial_\alpha X^k\partial_\beta X^j\ . \tag{3.3}$$

[2] The last relation in (2.18) follows from the definition $\gamma_{10} \equiv -\gamma_0\,\gamma_1\ldots\gamma_9$.

The equations of motion for Y^μ and X^k are given by

$$\partial_\beta\big(\sqrt{-h}h^{\alpha\beta}\partial_\alpha Y_\mu\big) = -\tilde{\Lambda}\, Y_\mu\,, \tag{3.4a}$$
$$\partial_\beta\big(\sqrt{-h}h^{\alpha\beta}\partial_\alpha X_k\big) = -\Lambda\, X_k\,, \tag{3.4b}$$

where the indexes of Y^μ are lowered and raised by $\eta_{\mu\nu}$. It is also useful to define the variables

$$Z_0 = Y^0 + iY^4 = \frac{R}{2}\,\sqrt{1+r^2}\,\mathrm{e}^{it}\,, \tag{3.5a}$$
$$Y^i = \frac{R}{2}\,r\,n^i\,, \qquad i=1,2,3\,, \tag{3.5b}$$
$$Z^i = X^{2i-1} + i\,X^{2i} = R\,\mu_i\,\mathrm{e}^{i\xi^i}\,, \qquad i=1,\dots,4\,, \tag{3.5c}$$

where the constraints (3.2) enforce $\vec{n}\cdot\vec{n}=1$ and $\sum_{i=1}^{4}{\mu_i}^2=1$; their equations of motion read,

$$\partial_\beta\big(\sqrt{-h}h^{\alpha\beta}\partial_\alpha Z_0\big) = -\tilde{\Lambda}\, Z_0\,, \tag{3.6a}$$
$$\partial_\beta\big(\sqrt{-h}h^{\alpha\beta}\partial_\alpha \vec{Y}\big) = -\tilde{\Lambda}\, \vec{Y}\,, \tag{3.6b}$$
$$\partial_\beta\big(\sqrt{-h}h^{\alpha\beta}\partial_\alpha Z^i\big) = -\Lambda\, Z^i\,. \tag{3.6c}$$

3.2 General ansatz

We now introduce the following ansatz,

$$\begin{aligned} t &= \omega_0\,\sigma^0\,, & r &= 0\,,\\ \mu_i &= \text{constant}\,, & \xi^i &= \omega_i\,\sigma^0 + m_i\,\sigma^1 + n_i\,\sigma^2 \equiv \frac{1}{2}\beta^i_\alpha\sigma^\alpha\,, \end{aligned} \tag{3.7}$$

where σ^1, σ^2 are 2π-periodic. Since m_i and n_i represent winding numbers, all of them must be integers; furthermore, for convenience we have introduced the compact notation, $\beta^i_0 = 2\,\omega_i$, $\beta^i_1 = 2\,m_i$, $\beta^i_2 = 2\,n_i$.[3]

Solutions with this structure were found in [15] in a particular gauge where $h_{01}=h_{02}=0$, $h_{00}=\text{const.}(h_{12}^2-h_{11}h_{22})$.[4] However, we will be later interested in a special class of solutions (called "non-collapsed membranes") for which this gauge choice is inconvenient. Therefore the analysis of solutions will be carried out in an arbitrary gauge.

The i index of β^i_α can be raised with the ξ^i part of the metric (2.7), i.e.,

$$\beta_{i,\alpha} \equiv {\mu_i}^2\,\beta^i_\alpha\,, \qquad \beta_{i,\alpha}\beta^i_\beta \equiv \sum_{i=1}^{4}\mu_i^2\beta^i_\alpha\beta^i_\beta\,. \tag{3.8}$$

[3] The index i in ω_i, m_i, n_i has been written as a *subindex* to avoid confusion with powers in the formulas containing specific values of i (e.g. we prefer to write m_2 instead of m^2).

[4] Generalizations of the solutions of [15] including non-constant μ_i were discussed in [19] (extending the integrable string σ models of [14] to membranes).

The world-volume metric becomes

$$h_{\alpha\beta} = \frac{R^2}{4}\left(\beta_{i,\alpha}\beta^i_\beta - \omega_0^2\,\delta_{\alpha,0}\delta_{\beta,0}\right)\,, \tag{3.9}$$

$$h = -\frac{R^6}{64}\left\{\omega_0^2\left[\left(\beta_{i,1}\beta^i_1\right)\left(\beta_{j,2}\beta^j_2\right) - \left(\beta_{i,1}\beta^i_2\right)^2\right] - \det{}_{\alpha,\beta}\left(\beta_{i,\alpha}\beta^i_\beta\right)\right\}\,. \tag{3.10}$$

The ansatz (3.7) includes momentum and winding around all four ξ^i angles. However, by performing a redefinition in the world-volume coordinates, we can reduce it to a problem with rotation in two planes only. Namely, defining

$$\begin{aligned}
\tilde\sigma^0 &= \sigma^0\,,\\
\tilde\sigma^1 &= \frac{1}{2}\left(\beta^1_0\,\sigma^0 + \beta^1_1\,\sigma^1 + \beta^1_2\,\sigma^2\right),\\
\tilde\sigma^2 &= \frac{1}{2}\left(\beta^3_0\,\sigma^0 + \beta^3_1\,\sigma^1 + \beta^3_2\,\sigma^2\right),
\end{aligned} \tag{3.11}$$

the ansatz (3.7) reduces to

$$\begin{aligned}
\xi^1 &= \tilde\sigma^1\,, \qquad & \xi^2 &= \tilde\omega_2\,\tilde\sigma^0 + \tilde m\,\tilde\sigma^1 + \tilde n'\,\tilde\sigma^2\,,\\
\xi^3 &= \tilde\sigma^2\,, \qquad & \xi^4 &= \tilde\omega_4\,\tilde\sigma^0 + \tilde m'\,\tilde\sigma^1 + \tilde n\,\tilde\sigma^2\,.
\end{aligned} \tag{3.12}$$

It should be noted that it is (locally) equivalent to the original (3.7) only if the following condition holds,

$$\beta^1_1\,\beta^3_2 - \beta^1_2\,\beta^3_1 \neq 0\,. \tag{3.13}$$

Because of the periodicity of the σ^1, σ^2 variables, the solutions are not globally equivalent in general. We recall that winding numbers must be integers for membranes in $AdS_4 \times S^7$ (and $\in \mathbb{Z}/k$ for membranes in $AdS_4 \times S^7/\mathbb{Z}_k$).

We will be interested in the particular case $\tilde m' = \tilde n' = 0$, i.e. in the solution

$$\begin{aligned}
\xi^1 &= \tilde\sigma^1\,, \qquad & \xi^2 &= \tilde\omega_2\,\tilde\sigma^0 + \tilde m\,\tilde\sigma^1\,,\\
\xi^3 &= \tilde\sigma^2\,, \qquad & \xi^4 &= \tilde\omega_4\,\tilde\sigma^0 + \tilde n\,\tilde\sigma^2\,.
\end{aligned} \tag{3.14}$$

Returning to the σ^α variables, (3.14) corresponds to the following choice in eq. (3.7),

$$\begin{aligned}
\vec\omega &= (\omega_1,\ \omega_2,\ \omega_3,\ \omega_4)\,,\\
\vec m &= (a,\ \alpha\,a,\ b,\ \beta\,b)\,,\\
\vec n &= (c,\ \alpha\,c,\ d,\ \beta\,d)\,.
\end{aligned} \tag{3.15}$$

if we make the identifications,

$$\tilde m \equiv \alpha\,, \qquad \tilde n \equiv \beta\,, \qquad \tilde\omega_2 \equiv \omega_2 - \alpha\,\omega_1\,, \qquad \tilde\omega_4 \equiv \omega_4 - \beta\,\omega_3\,. \tag{3.16}$$

Equations (3.11) then take the form

$$\begin{aligned}
\tilde\sigma^0 &\equiv \sigma^0\,,\\
\tilde\sigma^1 &\equiv \omega_1\,\sigma^0 + a\,\sigma^1 + c\,\sigma^2\,,\\
\tilde\sigma^2 &\equiv \omega_3\,\sigma^0 + b\,\sigma^1 + d\,\sigma^2\,,
\end{aligned} \tag{3.17}$$

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and the condition (3.13) for this equivalence to hold now reads $ad - bc \neq 0$. One has the option of considering $\tilde{m}$, $\tilde{n}$ integers in (3.14), or the solution (3.7), (3.15), with $\vec{m}$, $\vec{n}$ integers, giving rise to globally inequivalent solutions.

The ansatz (3.14) leads to the following values for β^i_α:

$$(\beta^1_\alpha) = (0,\ 2,\ 0)\,, \quad (\beta^2_\alpha) = (2\,\tilde{\omega}_2,\ 2\,\tilde{m},\ 0)\,, \quad (\beta^3_\alpha) = (0,\ 0,\ 2)\,, \quad (\beta^4_\alpha) = (2\,\tilde{\omega}_4,\ 0,\ 2\,\tilde{n})\,. \tag{3.18}$$

When $a\,d - b\,c = 0$, $\vec{m}$ results proportional to $\vec{n}$. More generally, whenever $\vec{m} = K\vec{n}$, we have $\beta^1_1\,\beta^3_2 - \beta^1_2\,\beta^3_1 = 0$ (or $ad - bc = 0$) and the change of coordinates (3.11) (or (3.17)) is not possible. Instead, it will be more convenient to introduce a new world-volume coordinate $\sigma = \sigma^2 + K\,\sigma^1$, exhibiting the fact that the configuration depends only on σ. This is the case when the M2 brane collapses to a string-like configuration.

3.3 Solving the conditions on the parameters

The equations of motion (3.6) impose some conditions on the parameters characterizing the solution. In order to solve these conditions for the ansatz (3.14), we first compute the inverse matrix $h^{\alpha\beta} = \frac{h_c^{\alpha\beta}}{h}$, where $h_c^{\alpha\beta}$ is the co-factor matrix of $h_{\alpha\beta}$. Its explicit expression is given in the appendix A (for clarity in the notation, in this section and in the appendix we will remove "tildes" from $\tilde{\omega}_2$, ,$\tilde{\omega}_4$, $\tilde{m}$, $\tilde{n}$).

The equations of motion (3.6) then reduce to

$$-{\omega_0}^2\,h_c^{00} = \sqrt{-h}\,\tilde{\Lambda}\,, \tag{3.19}$$

$$-\frac{1}{4}\,h_c^{\alpha\beta}\,\beta^i_\alpha\,\beta^i_\beta = \sqrt{-h}\,\Lambda\,, \qquad i = 1,\ldots,4\,. \tag{3.20}$$

While the first equation just fixes the value of $\tilde{\Lambda}$, the second one gives non-trivial conditions, since it must be satisfied for each $i = 1,\ldots,4$. One of the equations determines Λ and, generically, three independent conditions remain.

Using the expressions for $h_c^{\alpha\beta}$ given in the appendix A, equations (3.19)–(3.20) become

$$-\frac{\sqrt{-h}}{{\omega_0}^2\,R^4}\,\tilde{\Lambda} = ({\mu_1}^2 + {\mu_2}^2\,m^2)\,({\mu_3}^2 + {\mu_4}^2\,n^2)\,, \tag{3.21}$$

$$-\frac{4\sqrt{-h}}{\omega_0^2\,R^4}\,\Lambda = ({\mu_3}^2 + n^2\,{\mu_4}^2)\,\left({\mu_2}^2\,\left(\frac{2\,\omega_2}{\omega_0}\right)^2 - 1\right) + {\mu_3}^2\,{\mu_4}^2\,\left(\frac{2\,\omega_4}{\omega_0}\right)^2, \tag{3.22a}$$

$$-\frac{4\sqrt{-h}}{\omega_0^2\,R^4}\,\Lambda = ({\mu_3}^2 + n^2\,{\mu_4}^2)\,\left({\mu_1}^2\,\left(\frac{2\,\omega_2}{\omega_0}\right)^2 - m^2\right) + {\mu_3}^2\,{\mu_4}^2\,m^2\,\left(\frac{2\,\omega_4}{\omega_0}\right)^2, \tag{3.22b}$$

$$-\frac{4\sqrt{-h}}{\omega_0^2\,R^4}\,\Lambda = ({\mu_1}^2 + m^2\,{\mu_2}^2)\,\left({\mu_4}^2\,\left(\frac{2\,\omega_4}{\omega_0}\right)^2 - 1\right) + {\mu_1}^2\,{\mu_2}^2\,\left(\frac{2\,\omega_2}{\omega_0}\right)^2, \tag{3.22c}$$

$$-\frac{4\sqrt{-h}}{\omega_0^2\,R^4}\,\Lambda = ({\mu_1}^2 + m^2\,{\mu_2}^2)\,\left({\mu_3}^2\,\left(\frac{2\,\omega_4}{\omega_0}\right)^2 - n^2\right) + {\mu_1}^2\,{\mu_2}^2\,n^2\,\left(\frac{2\,\omega_2}{\omega_0}\right)^2, \tag{3.22d}$$

where the determinant h of $h_{\alpha\beta}$ is given by,

$$-\frac{4}{\omega_0{}^2\, R^6}\, h = ({\mu_1}^2 + m^2\, {\mu_2}^2)\,({\mu_3}^2 + n^2\, {\mu_4}^2) - {\mu_1}^2\, {\mu_2}^2\, ({\mu_3}^2 + n^2\, {\mu_4}^2) \left(\frac{2\,\omega_2}{\omega_0}\right)^2 - {\mu_3}^2\, {\mu_4}^2\, ({\mu_1}^2 + m^2\, {\mu_2}^2) \left(\frac{2\,\omega_4}{\omega_0}\right)^2 . \tag{3.23}$$

For generic[5] values of the μ_i's the relations (3.22) impose three conditions on the parameters. They can be solved explicitly in terms of a free variable z as follows,

$$m^2 = \frac{{\mu_1}^2}{{\mu_2}^2}\, \frac{z - {\mu_1}^2}{z - {\mu_2}^2}\,, \qquad n^2 = \frac{{\mu_3}^2}{{\mu_4}^2}\, \frac{z - {\mu_3}^2}{z - {\mu_4}^2}\,, \tag{3.24a}$$

$$\left(\frac{2\,\omega_2}{\omega_0}\right)^2 = \frac{1}{{\mu_2}^2}\, \frac{2\,z - {\mu_1}^2 - {\mu_2}^2}{z - {\mu_2}^2}\, \frac{1}{(3\,z - 1)(z - z_0)}\, \left(z^2 - z_2\, z + ({\mu_1}^2 + {\mu_2}^2)\, z_0\right)\,, \tag{3.24b}$$

$$\left(\frac{2\,\omega_4}{\omega_0}\right)^2 = \frac{1}{{\mu_4}^2}\, \frac{2\,z - {\mu_3}^2 - {\mu_4}^2}{z - {\mu_4}^2}\, \frac{1}{(3\,z - 1)(z - z_0)}\, \left(z^2 - z_4\, z + ({\mu_3}^2 + {\mu_4}^2)\, z_0\right)\,, \tag{3.24c}$$

where we have defined,

$$\begin{aligned} z_0 &\equiv C_0 \Big({\mu_1}^2\, {\mu_4}^2\, ({\mu_2}^2 + {\mu_3}^2) - {\mu_2}^2\, {\mu_3}^2\, ({\mu_1}^2 + {\mu_4}^2) \Big)\,, \\ z_2 &\equiv C_0 \Big({\mu_1}^2\, {\mu_4}^2\, (1 + {\mu_2}^2 - {\mu_4}^2) - {\mu_2}^2\, {\mu_3}^2\, (1 + {\mu_1}^2 - {\mu_3}^2) \Big)\,, \\ z_4 &\equiv C_0 \Big({\mu_1}^2\, {\mu_4}^2\, (1 + {\mu_3}^2 - {\mu_1}^2) - {\mu_2}^2\, {\mu_3}^2\, (1 + {\mu_4}^2 - {\mu_2}^2) \Big)\,, \end{aligned} \tag{3.25}$$

and $C_0 \equiv ({\mu_1}^2\, {\mu_4}^2 - {\mu_2}^2\, {\mu_3}^2)^{-1}$. They satisfy the relations,

$$z_2 + z_4 - 2\, z_0 = 1\,, \qquad z_2 - z_4 = {\mu_1}^2 + {\mu_2}^2 - {\mu_3}^2 - {\mu_4}^2\,. \tag{3.26}$$

For completeness, we also give the expression for the Lagrange multiplier parameters,

$$\begin{aligned} -\frac{\sqrt{-h}}{\omega_0{}^2\, R^4}\, \tilde{\Lambda} &= {\mu_1}^2\, {\mu_3}^2\, \frac{2\,z - {\mu_1}^2 - {\mu_2}^2}{z - {\mu_2}^2}\, \frac{2\,z - {\mu_3}^2 - {\mu_4}^2}{z - {\mu_4}^2}\,, \\ -\frac{4\sqrt{-h}}{\omega_0{}^2\, R^4}\, \Lambda &= -\frac{1}{3\,z - 1}\, {\mu_1}^2\, {\mu_3}^2\, \frac{2\,z - {\mu_1}^2 - {\mu_2}^2}{z - {\mu_2}^2}\, \frac{2\,z - {\mu_3}^2 - {\mu_4}^2}{z - {\mu_4}^2}\,. \end{aligned} \tag{3.27}$$

We find that the on-shell value of the determinant of the metric (3.23) is given by

$$-\frac{4}{\omega_0{}^2\, R^6}\, h = {\mu_1}^2\, {\mu_3}^2\, \frac{2\,z - {\mu_1}^2 - {\mu_2}^2}{z - {\mu_2}^2}\, \frac{2\,z - {\mu_3}^2 - {\mu_4}^2}{z - {\mu_4}^2}\, \frac{z}{3\,z - 1}\,. \tag{3.28}$$

Of particular interest is the case $z = 0$, because it characterizes a supersymmetric solution (see section 6.2). In this $h = 0$ case the membrane becomes tensionless. A similar phenomenon for supersymmetric rotating strings had been noticed in [17]. These M2 brane configurations with $h = 0$ are thus the precise higher dimensional analog of the rotating

[5] By generic we mean that all the μ_i's are non zero and different from each other.

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strings of [17]. When $z = 0$ the winding numbers and angular velocities are (up to signs) determined by the μ_i by the following relations:

$$m^2 = \frac{{\mu_1}^4}{{\mu_2}^4}\,, \quad n^2 = \frac{{\mu_3}^4}{{\mu_4}^4}\,, \quad \left(\frac{2\,\omega_2}{\omega_0}\right)^2 = \left(1+\frac{{\mu_1}^2}{{\mu_2}^2}\right)^2, \quad \left(\frac{2\,\omega_4}{\omega_0}\right)^2 = \left(1+\frac{{\mu_3}^2}{{\mu_4}^2}\right)^2, \tag{3.29}$$

where we have used equations (3.24). This solution is continuously connected with the $|z| = \infty$ solution, for which $h \neq 0$ and

$$m^2 = \frac{{\mu_1}^2}{{\mu_2}^2}\,, \qquad n^2 = \frac{{\mu_3}^2}{{\mu_4}^2}\,, \qquad \left(\frac{2\,\omega_2}{\omega_0}\right)^2 = \frac{2}{3\,{\mu_2}^2}\,, \qquad \left(\frac{2\,\omega_4}{\omega_0}\right)^2 = \frac{2}{3\,{\mu_4}^2}\,. \tag{3.30}$$

4 BPS bound from the superalgebra

In this section we use the superalgebra on the $\mathrm{AdS}_4 \times S^7$ vacuum to show that a solution that preserves a fraction of the supersymmetries must obey a simple bound. Our discussion follows the similar derivation given in [17] for $\mathrm{AdS}_5 \times S^5$. The $\mathrm{AdS}_4 \times S^7$ vacuum has the isometry superalgebra $OSp(4|8)$. The bosonic symmetry is $\mathrm{SO}(2,3) \times \mathrm{SO}(8)$. The supercharges are 32 Majorana spinors which under the $\mathrm{SO}(2,3) \times \mathrm{SO}(8)$ subgroup of the 11d Lorentz group $\mathrm{SO}(1,10)$ decompose as 4-component Majorana spinors Q_a, with $a = 1,\dots,8$, transforming in the spinorial $\mathbf{8_s}$ representation of SO(8) (more precisely, $spin(8)$). Let us denote by $\tilde{\gamma}_\mu$ (in this section $\mu,\nu = 0,1,2,3$) the 4×4 four-dimensional Dirac matrices for AdS_4. The anticommutators are

$$\{Q_a,\ Q_b\} = C\left[\left(\tilde{\gamma}_\mu\, P^\mu + \frac{1}{2}\,\tilde{\gamma}_{\mu\nu}\, M^{\mu\nu}\right)\delta_{ab} + \mathbb{I}\,\hat{B}_{ab}\right], \tag{4.1}$$

where C is the charge conjugation matrix ($C = \tilde{\gamma}^0$ for the real Majorana representation), P^μ, $M^{\mu\nu}$ are the charges in AdS_4 , and $\hat{B}_{ab}$ is a real antisymmetric matrix of $spin(8)$ charges. For our solutions, the only non-vanishing charges are the energy P^0 and the angular momenta $J_1,\dots,J_4$. These last ones are eigenvalues of the Cartan generators of SO(8) in the vector representation $\mathbf{8_v}$. Using the standard relation $\hat{B}_{ab} = \frac{1}{4}\hat{\gamma}^{ij}_{ab}B_{ij}$, where $\{\hat{\gamma}^i, i = 1,\dots,8\}$ are the gamma matrices in the spinorial $\mathbf{8_s}$ representation, and putting B_{ij} in block-diagonal form by means of a SO(8) transformation, we have

$$B_{ij} = \mathrm{diag}\left[\begin{pmatrix} 0 & J_1 \\ -J_1 & 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 & J_4 \\ -J_4 & 0 \end{pmatrix}\right], \tag{4.2}$$

and similarly for $\hat{B}_{ab}$, with $\hat{b}_1,\dots,\hat{b}_4$ instead of J_i. The non-vanishing elements of $\hat{B}_{ab}$ are related to the J_i's by

$$\begin{aligned} \hat{b}_1 &= \frac{1}{2}(-J_1+J_2+J_3+J_4)\,, \qquad & \hat{b}_2 &= \frac{1}{2}(+J_1-J_2+J_3+J_4)\,, \\ \hat{b}_3 &= \frac{1}{2}(+J_1+J_2-J_3+J_4)\,, \qquad & \hat{b}_4 &= \frac{1}{2}(+J_1+J_2+J_3-J_4)\,. \end{aligned} \tag{4.3}$$

The anticommutation relations then become

$$\{Q_a,\ Q_b\} = \mathbb{I}\,\delta_{ab}\,P^0 + \tilde{\gamma}^0\,\hat{B}_{ab}\,. \tag{4.4}$$

Since $(\tilde{\gamma}^0)^2 = -1$, the eigenvalues of $\tilde{\gamma}^0 \hat{B}$ are $\pm\hat{b}_i$. Therefore, the eigenvalues of the anticommutator matrix are $P^0 \pm \hat{b}_i$, $i = 1, \ldots, 4$. In any unitary representation the matrix $\{Q_a,\ Q_b\}$ is definite positive, thus the BPS bound is

$$P^0 \geq \hat{b}_{\max} \ . \tag{4.5}$$

where $\hat{b}_{\max}$ is the maximum of $\pm\hat{b}_i$.

P^0 generates translations in the time t. For the membranes considered in this paper lying at $r = 0$, the proper time is given by $d\tau = \frac{R}{2}dt$, see (2.1). Therefore their energies E are related to P^0 by $E = 2P^0/R$. Defining $\eta_i = \mathrm{sgn}(J_i)$, the signs are subject to the condition $\eta_1\eta_2\eta_3 = -\eta_4$. This implies that, for these membranes, $\hat{b}_{\max}$ is nothing but $\frac{1}{2}\sum_{i=1}^4 |J_i|$. Thus the energies of our membrane solutions are subject to the bound

$$E \geq \frac{1}{R}\sum_{i=1}^{4} |J_i| \ . \tag{4.6}$$

When the bound is saturated, the matrix of anticommutators have some zero eigenvalues, implying that some fraction of supersymmetry is preserved.

When three or more J_i are non-vanishing and generic,[6] there is only one vanishing eigenvalue and the corresponding state saturating the bound preserves 1/8 of the supersymmetries. When two J_i are non-zero and generic, there are two vanishing eigenvalues and the corresponding state preserves 1/4 of the supersymmetries. Finally, states with only one non-zero J_i have four vanishing eigenvalues and the solution preserves 1/2 of the supersymmetries.

5 Energy and angular momenta

5.1 General formulas

According to Noether's theorem, if

$${}^{\epsilon}X^{\mu} = X^{\mu} + \epsilon\, \delta X^{\mu} + o(\epsilon^2) \ . \tag{5.1}$$

is a continuum transformation with parameter ϵ such that $S[{}^{\epsilon}X, h] = S[X, h]$, then

$$J^{\alpha} = \delta X^{\mu} \left.\frac{\partial \mathcal{L}}{\partial \partial_{\alpha} X^{\mu}}\right|_{\text{on-shell}} , \tag{5.2}$$

is a conserved current,

$$\nabla_{\alpha} J^{\alpha} = 0 \, , \tag{5.3}$$

and, therefore,

$$Q \equiv \int \mathrm{d}\sigma^1 \mathrm{d}\sigma^2\ J^0 = -T_2 \int \mathrm{d}\sigma^1 \mathrm{d}\sigma^2\ \sqrt{-h}\ h^{0\alpha}\ G_{\mu\nu}(X)\ \delta X^{\mu}\ \partial_{\alpha} X^{\nu} \, , \tag{5.4}$$

[6] When some J_i have coincident values, some $\hat{b}_i$ will be equal to each other, implying the possibility of enhancement of supersymmetry. However, it is easy to see that this possibility is not realized our membrane solutions subject to the condition $\eta_1\eta_2\eta_3 = -\eta_4$.

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is a conserved quantity

$$\frac{\mathrm{d}Q}{\mathrm{d}\sigma^0} = 0 \ . \tag{5.5}$$

If we apply this standard procedure to isometries of the background (and hence symmetry transformations) we can define the following conserved charges.

Energy: $\frac{R}{2}\,\delta X^0 = \frac{R}{2}\,\delta t = -1$.

$$E = V_2\, T_2\, R\, \frac{\omega_0}{2}\, \frac{h_c^{00}}{\sqrt{-h}} \ , \qquad V_2 \equiv \int \mathrm{d}\sigma^1 \mathrm{d}\sigma^2 = 4\pi^2 \ . \tag{5.6}$$

Angular momenta: $\delta\xi^i = 1\ , \forall i$.

$$J_i = V_2\, T_2\ R^2\, \frac{h_c^{0\alpha}}{\sqrt{-h}}\, \frac{\beta^i_\alpha}{2}\, {\mu_i}^2 = E\, \frac{R}{\omega_0}\, {\mu_i}^2 \left(\beta^i_0 + \frac{h_c^{01}}{h_c^{00}}\,\beta^i_1 + \frac{h_c^{02}}{h_c^{00}}\,\beta^i_2\right) \ . \tag{5.7}$$

5.2 Energy and momenta of non-collapsed membranes

Evaluating these formulas on our family of solutions (3.24)

$$E = V_2\, T_2\, R^2\, \mu_1\, \mu_3 \left(\frac{2\,z - {\mu_1}^2 - {\mu_2}^2}{z - {\mu_2}^2}\, \frac{2\,z - {\mu_3}^2 - {\mu_4}^2}{z - {\mu_4}^2}\, \frac{3\,z - 1}{z}\right)^{1/2} \ , \tag{5.8a}$$

$$\eta_1\, J_1 = R\, E\, \mu_1 \left(\frac{z - {\mu_1}^2}{(3\,z-1)(z-z_0)}\, \frac{z^2 - z_2\,z + ({\mu_1}^2 + {\mu_2}^2)\,z_0}{2\,z - {\mu_1}^2 - {\mu_2}^2}\right)^{1/2} \ , \tag{5.8b}$$

$$\eta_2\, J_2 = R\, E\, \mu_2 \left(\frac{z - {\mu_2}^2}{(3\,z-1)(z-z_0)}\, \frac{z^2 - z_2\,z + ({\mu_1}^2 + {\mu_2}^2)\,z_0}{2\,z - {\mu_1}^2 - {\mu_2}^2}\right)^{1/2} \ , \tag{5.8c}$$

$$\eta_3\, J_3 = R\, E\, \mu_3 \left(\frac{z - {\mu_3}^2}{(3\,z-1)(z-z_0)}\, \frac{z^2 - z_4\,z + ({\mu_3}^2 + {\mu_4}^2)\,z_0}{2\,z - {\mu_3}^2 - {\mu_4}^2}\right)^{1/2} \ , \tag{5.8d}$$

$$\eta_4\, J_4 = R\, E\, \mu_4 \left(\frac{z - {\mu_4}^2}{(3\,z-1)(z-z_0)}\, \frac{z^2 - z_4\,z + ({\mu_3}^2 + {\mu_4}^2)\,z_0}{2\,z - {\mu_3}^2 - {\mu_4}^2}\right)^{1/2} \ . \tag{5.8e}$$

where we have introduced the signs of ω_2, ω_4, m, n in the following way,

$$\mathrm{sgn}(\omega_2) \equiv \eta_2\,, \qquad \mathrm{sgn}(\omega_4) \equiv \eta_4\,, \qquad \mathrm{sgn}(m) \equiv -\eta_2\,\eta_1\,, \qquad \mathrm{sgn}(n) \equiv -\eta_4\,\eta_3\,, \tag{5.9}$$

so that $\mathrm{sgn}(J_i) = \eta_i$.

In the limit $z \to 0$ both E and J_i tend to infinity. It is straightforward to show that in this limit,

$$J_i = \eta_i\, R\, E\, {\mu_i}^2 \qquad \Longrightarrow \qquad E = \frac{1}{R}\sum_{i=1}^{4} |J_i| \ . \tag{5.10}$$

This simple relation is due to the fact that in this limit the solution becomes supersymmetric, as explained in section 4 and will be seen more explicitly in section 6.

On the other hand, in the opposite limit $|z| = \infty$, one finds the solution with $J_i = \eta_i\, \frac{R}{\sqrt{6}}\, E\, \mu_i$, giving

$$E^2 = \frac{6}{R^2} \sum_{i=1}^{4} {J_i}^2 \ . \tag{5.11}$$

There is no preserved supersymmetry for this solution. The general relation between E and J_i for solutions with arbitrary z is given in appendix A, for completeness.

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5.3 Energy and momenta of collapsed membranes

Let us now consider the collapsed membrane configurations with $\vec{m} = K\,\vec{n}$. In this case, the expressions (5.6), (5.7) for E and J^α become ambiguous and need a proper regularization. The same ambiguity occurs for the BMN string if one attempts to compute the energy and angular momentum using the Nambu-Goto action. In this case, the solution describing the BMN state is $X^0 = \omega_0\,\tau$, $\phi = \omega_0\,\tau$, where ϕ is an angle of the S^5 sphere. The proper way to do the calculation is, as in [20], to use the Polyakov action in the conformal gauge, and then compute E and J (obtaining $E \propto J$). For membranes, there is no possibility of a conformal gauge. The closer analog is the gauge $h_{01} = h_{02} = 0$ and $h_{00} = -(h_{11}\,h_{22} - {h_{12}}^2)$. The formulas (5.6), (5.7) for the energy and angular momentum in this gauge reduce to

$$E = V_2\,T_2\,\frac{R}{2}\,\omega_0\,, \qquad J_i = V_2\,T_2\,R^2\,\mu_i^2\,\omega_i\ . \tag{5.12}$$

In addition, for the collapsed membrane with $\vec{m} = K\vec{n}$, $g = 0$ and the constraint $h_{00} = -g$ implies the relation

$$\omega_0 = 2\sqrt{\sum_{i=1}^{4} \mu_i^2\,\omega_i^2}\ . \tag{5.13}$$

Since $\omega_0 \neq 0$, at least one μ_i and ω_i must be non-vanishing. Taking $\mu_1 \neq 0$, $\omega_1 \neq 0$, the equations of motion then imply the additional relation

$${\omega_i}^2 = {\omega_1}^2 = -\Lambda\,, \qquad \forall i \text{ such that } \mu_i \neq 0\ . \tag{5.14}$$

It follows that $\omega_0 = 2\,|\omega_1|$. This agrees with the general formulas of [15] particularized to the case $\vec{m} = K\vec{n}$. Therefore

$$E = \frac{J}{R}\,, \qquad J = \sum_i |J_i|\ . \tag{5.15}$$

In addition, the constraint associated with the gauge choice $h_{01} = h_{02} = 0$ imposes the condition

$$\sum_{i=1}^{4} m_i\,J_i = 0\ . \tag{5.16}$$

The derivation of the previous formulas implies dealing with membranes with null world-volume, i.e. $h = 0$, for which classical methods are not, in general, justified. Indeed, these membranes can be more properly viewed as the limit of large angular momentum of general non-collapsed, regular membranes of the form (3.7). This is obviously the case as can be explicitly seen from the general formulas given in section 4 of [15], where the large J limit indeed leads to the conditions (5.13), (5.14) and (5.15) (while the condition (5.16) holds for any finite J). In general, one finds [15] $E = J/R + O(1/J)$.

6 Supersymmetry conditions for the solutions

6.1 Supersymmetry equations

We shall now investigate the configurations of the form (3.7) which preserve some fraction of supersymmetry. A configuration preserves a supersymmetry for every independent Killing spinor ϵ defined in (2.16) that satisfies

$$\Gamma_\kappa\, \epsilon = \pm\epsilon\,, \qquad \Gamma_\kappa = \frac{1}{3!\sqrt{-h}}\epsilon^{abc}\partial_a X^\mu \partial_b X^\nu \partial_c X^\rho \Gamma_{\mu\nu\rho}\,, \tag{6.1}$$

where Γ_κ is the κ-symmetry matrix, the Gamma matrices are given by (2.15), and $+1(-1)$ stands for the M2 (anti) brane. Substituting the ansatz (3.7) we find

$$\Gamma_\kappa = \frac{R^3}{8\sqrt{-h}}\mu_i\, \mu_j\, \beta_1^i \beta_2^j \Big(\omega_0\, \gamma_0 + \mu_k \beta_k^0 \gamma_{k+6}\Big)\gamma_{i+6}\, \gamma_{j+6}\ . \tag{6.2}$$

where summation over i, j is understood and k indexes between 1 and 4.

Using (2.16), we find that the Killing spinors of $AdS_4 \times S^7$ must satisfy

$$\mathcal{M}^{-1}\, \Gamma_\kappa\, \mathcal{M}\, \epsilon_0 = \pm\, \epsilon_0\ . \tag{6.3}$$

After some algebra, the full supersymmetry equations reduce to[7]

$$\begin{aligned} &\left\{\sum_{i<j}\gamma_0\gamma_{i+6}\gamma_{j+6}\Big(\omega_0\mu_i\mu_j\beta_{12}^{ij} - \mu_i\mu_j\mu_k^2\beta^{ijk}O_k\Big)M_t^2M_i^2M_j^2\right.\\ &\left.-\gamma_{7,8,9,10}\sum_{ijkl}\varepsilon_{ijkl}\beta^{ijk}\mathbb{X}_l\prod_{k=1}^{4}\mu_kM_k^2\right\}\epsilon_0 \\ &\qquad = -\sqrt{\omega_0^2\sum_{i<j}\mu_i^2\mu_j^2(\beta_{12}^{ij})^2 - \sum_{i<j<k}\mu_i^2\mu_j^2\mu_k^2(\beta^{ijk})^2}\ \epsilon_0\,, \end{aligned} \tag{6.4}$$

where

$$\begin{aligned} O_k &= \hat\gamma\ \gamma_0\ \mathbb{X}_k\,,\\ O_4 &= -O_1\, O_2\, O_3\,, \end{aligned} \tag{6.5}$$

the $\mathbb{X}_k$ have been defined in (2.18), and

$$\begin{aligned} \beta_{\alpha\beta}^{ij} &\equiv \beta_\alpha^i\beta_\beta^j - \beta_\alpha^j\beta_\beta^i\,,\\ \beta^{ijk} &\equiv \beta_0^i\ \beta_{12}^{jk} + \beta_0^j\ \beta_{12}^{ki} + \beta_0^k\beta_{12}^{ij}\ . \end{aligned} \tag{6.6}$$

Equation (6.4) is highly complicated in general. However, for our ansatz (3.7), (3.15), gets simplified in a striking way. In particular, it is easy to check that all the β_{12}^{ij} either vanish or are proportional to

$$N = ad - bc\ . \tag{6.7}$$

This implies that both terms of (6.4) are proportional to N. On the face of it, it might seem that if $N = 0$ then the supersymmetry condition (6.4) is trivially satisfied for all 32 spinors ϵ_0. However, the $N = 0$ case is rather subtle, because in this case $\vec{m}$ is proportional to $\vec{n}$ and the M2 brane collapses to a string-like configuration, as explained at the end of section 3.2. In this case the equation (6.1) becomes singular and cannot be used. We will return to this case in section 6.3.

[7]This algebra requires commuting $\mathcal{M}$ with Γ matrices. Useful relations can be found in the appendix B of [12].

6.2 Supersymmetry of the non-collapsed membranes

We first investigate the supersymmetry conditions for $N \neq 0$, for generic values of the μ_i's. Let η_k denote the eigenvalues of the O_k operators,

$$O_k\, \epsilon_0 = \eta_k\, \epsilon_0 \ , \qquad k = 1, 2, 3\ . \tag{6.8}$$

Since $O_k^2 = 1$, the eigenvalues are just equal to ± 1. This leads to only three independent conditions, since $\eta_4 = -\eta_1\eta_2\eta_3$ (see equation (6.5)). With no loss of generality we can set $\eta_1 = \eta_2 = \eta_3 = 1$, $\eta_4 = -1$, since the sign of any η_i can be reversed by a coordinate redefinition $\xi^i \to -\xi^i$. Let us start by fixing,

$$\alpha = -\frac{\mu_1}{\mu_2}\,, \qquad\qquad \beta = \frac{\mu_3}{\mu_4}\,, \tag{6.9}$$

By using (6.8), the supersymmetry condition (6.4) leads to two equations

$$\mu_1^2\,\omega_1 + \mu_2^2\,\omega_2 = \frac{1}{2}\ \omega_0\ (\mu_1^2 + \mu_2^2)\,, \tag{6.10a}$$

$$\mu_3^2\,\omega_3 - \mu_4^2\,\omega_4 = \frac{1}{2}\ \omega_0\ (\mu_3^2 + \mu_4^2)\ . \tag{6.10b}$$

Note that these equations only restrict the possible values of the parameters, but they do not imply any condition on the spinor. Therefore equations (6.10) do not reduce the number of supersymmetries. Once (6.10) are imposed on the parameters, both sides of the supersymmetry equation (6.4) become identically zero. In terms of the coordinates $\tilde{\sigma}^\alpha$, the solution takes the simple form

$$\begin{aligned} t &= \omega_0\,\tilde{\sigma}^0\,, & r &= 0\,, & \mu_i &= \text{constant},\\ \xi^1 &= \tilde{\sigma}^1\,, & \xi^2 &= \tilde{\omega}_2\,\tilde{\sigma}^0 + \tilde{m}\,\tilde{\sigma}^1\,, & &\\ \xi^3 &= \tilde{\sigma}^2\,, & \xi^4 &= \tilde{\omega}_4\,\tilde{\sigma}^0 + \tilde{n}\,\tilde{\sigma}^2\ . & & \end{aligned} \tag{6.11}$$

with

$$\tilde{\omega}_2 = \frac{1}{2}\,\omega_0\,\left(1 + \frac{\mu_1^2}{\mu_2^2}\right)\,, \qquad \tilde{\omega}_4 = -\frac{1}{2}\,\omega_0\,\left(1 + \frac{\mu_3^2}{\mu_4^2}\right)\,, \qquad \tilde{m} = -\frac{\mu_1^2}{\mu_2^2}\,, \qquad \tilde{n} = +\frac{\mu_3^2}{\mu_4^2}\ . \tag{6.12}$$

On shell (i.e. upon use of (3.24)), this M2 brane has a singular induced metric, $h = 0$. Nonetheless, it should be noted that the membrane is regular, in particular, it is not collapsed to a string, despite the fact that the induced world-volume metric has vanishing determinant $h = 0$. The phenomenon is similar to the one found for strings in [17]. The interpretation is that these configurations describe tensionless membranes, since the world-volume is null. Physically, it means that, for these membranes, the energy due to the tension is negligible compared to the energy due to rotation (see also section 5).

In conclusion, the M2 brane configuration (6.11) is supersymmetric for Killing spinors satisfying the three conditions (6.8). Therefore our solution preserves 1/8 of the supersymmetries of the background. Furthermore, we note that the solution is just the $z = 0$ solution to the equations of motion given in (3.29).

The number of supersymmetries can also be deduced from the BPS algebra. For generic values of μ_i's, the bound (4.6) is saturated with the four J_i non zero and different from each other, as shown in section 5.2, see (5.10). In this case the 8×8 matrix $\{Q_a, Q_b\}$ has a unique zero eigenvalue, hence only 1/8 of the supersymmmetries is preserved, in agreement with the above counting using the Γ_κ matrix.

6.3 Supersymmetry of the collapsed membranes

In terms of the new world-volume coordinate $\sigma \equiv \sigma^2 + K\,\sigma^1$, the solution for the M2 brane collapsed to a string is obtained from the ansatz (3.7) by simply setting $n_i = 0$. This gives

$$\begin{aligned} &t = \omega_0\,\tau\,, \qquad\qquad r = 0\,, \\ &\mu_i = \text{constant}\,, \quad \xi^i = \omega_i\,\tau + m_i\,\sigma \equiv \frac{1}{2}\,\beta^i_a\,\sigma^a\,, \end{aligned} \tag{6.13}$$

where $\sigma^a = (\sigma^0, \sigma^1) \equiv (\tau, \sigma)$, $a = 0, 1$.

The simplest way to study the supersymmetry of the collapsed membrane configuration is from the supersymmetry algebra. In section 5.3 we have seen that these configurations saturate the BPS bound (4.6), and therefore they are all supersymmetric. The preserved fraction of supersymmetries depends on how many J_i are different from zero:

- In the case of rotation in four planes with generic J_i's non-zero, the 8×8 matrix $\{Q_a, Q_b\}$ has only one zero eigenvalue. As a result, the solution preserves 1/8 of the supersymmetries.

- In the case of rotation in three planes, only one of the J_i vanishes, say J_4. Generically, the $\hat{b}_i$ given in (4.3) are still different from each other and as a result the matrix $\{Q_a, Q_b\}$ has still only one zero eigenvalue. This solution also preserves 1/8 of the supersymmetries.

- In the case of rotation in two planes, two of the J_i vanish, say J_3, J_4. From (4.3) we obtain $\hat{b}_1 = -\hat{b}_2$ and $\hat{b}_3 = \hat{b}_4$. It is easy to see that in this case the matrix $\{Q_a, Q_b\}$ has two zero eigenvalues, coming from $E \pm \frac{2}{R}\hat{b}_{3,4}$ or $E \pm \frac{2}{R}\hat{b}_1$, $E \mp \frac{2}{R}\,\hat{b}_2$, according to the signs of J_1, J_2. This solution preserves 1/4 of the supersymmetries.

- Finally, in the case of rotation in one plane, taking e.g. $J_2 = J_3 = J_4 = 0$, there are four vanishing eigenvalues when the BPS bound is saturated. The membrane preserves 1/2 of the supersymmetry. However, in this case the constraint (5.16) $\sum_{i=1}^4 m_i J_i = 0$ implies that $m_1 = 0$: the membrane collapses to a point. This is a BMN state.

The case of the M2 brane (6.13) collapsed to a string-like configuration the κ-symmetry matrix Γ_κ of the M2 brane is singular and cannot be used to determine the unbroken supersymmetries. The same problem exists for strings collapsing to a point, like in the BMN solution [5, 20], representing a collapsed string moving around the equator of S^5 at the speed of light; the Γ_κ matrix of the string is singular but one can use the supersymmetry algebra in a similar way as we did above to show that the solution preserves 1/2 of the supersymmetries (see e.g. [17]).

In the present case, since the membrane is collapsed to a string, one may try to determine the unbroken supersymmetries by using the Γ_κ matrix corresponding to an effective string. In appendix B we show that this approach reproduces the correct number of supersymmetries obtained above from the supersymmetry algebra.

7 Generalization to $\mathrm{AdS}_4 \times S^7/\mathbb{Z}_k$

The supersymmetric M2 brane configurations described in the previous sections admit a straightforward generalization to the case of $\mathrm{AdS}_4 \times S^7/\mathbb{Z}_k$. As explained above, the $\mathbb{Z}_k$ orbifold acts on the ξ_i angles by identification $\xi_i \sim \xi_i + 2\pi/k$. The spectrum on $\mathrm{AdS}_4 \times S^7/\mathbb{Z}_k$ is obtained by the projection of the original spectrum on $\mathbb{Z}_k$ invariant states. This leads to the following quantization conditions on the winding numbers:

$$m_i,\ n_i \in \mathbb{Z}/k\ . \tag{7.1}$$

Dimensional reduction of $\mathrm{AdS}_4 \times S^7/\mathbb{Z}_k$ along the y coordinate gives the $\mathrm{AdS}_4 \times CP^3$ space (see section 2). Finding novel supersymmetric states in this space is of particular interest given the connection with ABJM theory. To proceed, we recall that y is the diagonal part of the four ξ_i angles,

$$y = \frac{1}{4}\left(\xi_1 + \xi_2 + \xi_3 + \xi_4\right)\ . \tag{7.2}$$

For our general ansatz (3.7) this gives

$$y = \omega_y\,\sigma^0 + m_y\,\sigma^1 + n_y\,\sigma^2\,, \tag{7.3}$$

where we have defined

$$\omega_y = \frac{1}{4}\sum_i \omega_i\,, \qquad m_y = \frac{1}{4}\sum_i m_i\,, \qquad n_y = \frac{1}{4}\sum_i n_i\ . \tag{7.4}$$

The other coordinates $\psi, \varphi_1,\ \varphi_2$ in eq. (2.10) have a similar $\sigma^0,\ \sigma^1,\ \sigma^2$ dependence. As pointed out in [15], these type of configurations in the generic case correspond to non-perturbative objects in the type II string theory. Generally, in ten dimensions these configurations represent bound states of D0 branes, D2 branes and rotating circular fundamental strings. The D0 brane charge arises from the momentum in the y direction, $P_y = k\,\omega_y$. Because the circles $\psi, \varphi_1, \varphi_2$ are contractible, the net D2 brane and fundamental string charges are zero (just like the fundamental strings of [14]).

Consider in particular the 1/8 supersymmetric non-collapsed M2 brane solution (6.11). In this case

$$4\,y = (\omega_2 + \omega_4)\,\sigma^0 + (1+m)\,\sigma^1 + (1+n)\,\sigma^2\,, \tag{7.5}$$

where we removed tildes. Using (6.12), we see that D0 brane charge $P_y = \omega_0\,k\,(\mu_1^2\mu_4^2 - \mu_3^2\mu_2^2)/(8\mu_2^2\mu_4^2)$ is determined in terms of angles μ_i representing the location of the bound state system.

Consider now the collapsed membrane configurations of section 6.3. They are of the form

$$4\, y_{\text{coll.}} = \omega_y\, \sigma^0 + n_y\, \sigma^2\,, \tag{7.6}$$

where we now use σ^2, instead of σ, to avoid possible confusion with the world-sheet string coordinate σ of type IIA string theory. The other coordinates $\psi, \varphi_1,\ \varphi_2$ depend only on σ^0 and σ^2 as well. The configuration has non-vanishing D0 brane charge. To see this explicitly, we recall that another consequence of the orbifold projection is that the momentum along the y is quantized as $J_1 + J_2 + J_3 + J_4 = kp$, for p units of D0 brane charge (see related discussion in [12]). There are two cases to be distinguished:

a) $n_y = 0$. In this case the string-shaped membrane is not wrapped around the eleven dimensional circle y. The other coordinates $\psi, \varphi_1, \varphi_2$ will generically depend on σ_2, which, in this particular case, can be identified with the string world-sheet coordinate. The configuration then represents a bound state system of p D0 branes and fundamental strings with vanishing total charge.

b) $n_y \neq 0$. In this case the string-shaped membrane is now wrapped around the eleven dimensional circle y. As a result, upon reduction, the configuration does not contain any fundamental string, but it has p units of D0 brane charge.

It would be interesting to identify the dual BPS operators of ABJM three dimensional $\mathcal{N} = 6$ Chern-Simons theory, both for the collapsed membranes and for the 1/8 supersymmetric M2 brane (6.11). In general, these operators have conformal dimension $kp/2$ and (like the configurations of [13]) are to be given in terms of configurations involving non-abelian degrees of freedom in some non-trivial way.

8 Giant diabolo

8.1 BPS equation

In this section we study a different class of supersymmetric membranes that also extend to the AdS_4 part of the background. It is convenient to introduce cylindrical coordinates. The metric and three-form become

$$\begin{aligned} \mathrm{d}s^2 &= \frac{R^2}{4}\left\{-\left(1+z^2+\rho^2\right)\mathrm{d}t^2 + \frac{(z\,\mathrm{d}z+\rho\,\mathrm{d}\rho)^2}{(z^2+\rho^2)(1+z^2+\rho^2)} + \frac{(z\,\mathrm{d}\rho-\rho\,\mathrm{d}z)^2}{z^2+\rho^2} + \rho^2\mathrm{d}\varphi^2\right\} \\ &\quad + R^2\left\{\mathrm{d}\alpha^2 + \cos^2\alpha\,\mathrm{d}\beta^2 + \cos^2\alpha\,\cos^2\beta\,\mathrm{d}\gamma^2 + \sum_{i=1}^{4}\mu_i^2\,\mathrm{d}\xi_i^2\right\} \\ C^{(3)} &= \frac{R^3}{8}\,\rho\, dt\wedge(z\, d\rho - \rho\, dz)\wedge d\varphi. \end{aligned} \tag{8.1}$$

The ansatz is as follows

$$\begin{aligned} t &= \sigma^0 \quad , \qquad z = \sigma^2\,, \\ \varphi &= \alpha_0\,\sigma^0 + \alpha_1\,\sigma^1 + \alpha_2\,\sigma^2\,, \\ \xi^i &= m_i\,\left(s_0\,\sigma^0 + s_1\,\sigma^1 + s_2\,\sigma^2\right)\,, \\ \rho &= \rho(\sigma^2)\ . \end{aligned} \tag{8.2}$$

We shall derive the BPS equations using again the condition (6.1) on the background spinors. We first decompose the κ-symmetry matrix in two factors

$$\Gamma = -\tilde{\gamma}\,\tilde{\Gamma}\,, \qquad \tilde{\gamma} \equiv \frac{f'\,\gamma_1 + (z\rho' - \rho)\,\gamma_2}{r\,(1+\rho'^2 - f'^2)^{1/2}}\,, \qquad \tilde{\Gamma} \equiv \frac{1}{\sqrt{-h}}\left(\sum_{i=1}^{4}\delta_i + \tilde{\delta}\right)\,, \tag{8.3}$$

where $f^2(z) \equiv 1 + \rho^2(z) + z^2$, and

$$\delta_1 = \frac{1}{2}\alpha_1\rho f\gamma_{03}, \qquad \delta_2 = s_1 f\gamma_0 \sum_{i=1}^{4}\mu_i m_i \gamma_{i+6}, \qquad \delta_3 = (\alpha_0 s_1 - \alpha_1 s_0)\rho\gamma_3 \sum_{i=1}^{4}\mu_i m_i\gamma_{i+6}, \tag{8.4a}$$

$$\tilde{\delta} = (\alpha_1 s_2 - \alpha_2 s_1)\frac{\rho f}{(1+\rho'^2 - f'^2)^{1/2}}\,\gamma_{30}\,\tilde{\gamma}\,\sum_{i=1}^{4}\mu_i m_i \gamma_{i+6}\,. \tag{8.4b}$$

An important feature of these matrices is that $\tilde{\gamma}$ and $\tilde{\Gamma}$ do not commute unless $\tilde{\delta} = 0$. We will assume $\tilde{\delta} = 0$ in order to get an analytic solution. Thus we take

$$s_i = a\,\alpha_i\,, \qquad i = 1, 2\,, \tag{8.5}$$

with a arbitrary. Once this condition is implemented, our ansatz becomes equivalent to the ansatz considered in [12] using spherical coordinates. The cylindrical coordinates are more convenient to exhibit how the various geometries are realized for different values of the parameters. Some of the geometries shown here are novel.

The supersymmetry condition is

$$\left[\mathcal{M}^{-1}\tilde{\gamma}\mathcal{M}\;\mathcal{M}^{-1}\tilde{\Gamma}\mathcal{M} + \epsilon\right]\epsilon_0 = 0\,. \tag{8.6}$$

with $\epsilon = +1(-1)$ for the (anti) $M2$ brane. In order to cancel out terms proportional to $M_i^2 M_j^2$, for $i \neq j = 1,2,3,4$, we demand

$$\begin{aligned} m_i &= e_i\, m\,, \quad e_i^2 = 1\,, \qquad m > 0\,, \qquad && i = 1,2,3,4\,,\\ \mathbb{X}_i\,\mathbb{X}_4\,\epsilon_0 &= -e_i\, e_4\, \epsilon_0\,, && i = 1,2,3\,. \end{aligned} \tag{8.7}$$

Note that we can impose these conditions on the spinor since $\mathbb{X}_i^2 = -1$ and $[\mathbb{X}_i,\ \mathbb{X}_j] = 0$. However, due to the relation $\prod_{i=1}^{3}(\mathbb{X}_i\,\mathbb{X}_4) = +1$ (see 2.18), we have that $\prod_{i=1}^{4} e_i = -1$. Furthermore, this also implies that there are just two independent constraints in (8.7). The signs e_i can be reabsorbed into a redefinition of ξ^i, and m can be absorbed by the s_α's. Therefore in what follows we set $m_1 = m_2 = m_3 = -m_4 = +1$. We omit some details of the computation, which is straightforward, albeit tedious. The resulting conditions turn out to be

$$\gamma_{0,10}\,\epsilon_0 = \eta_1\,\epsilon_0\,, \qquad \gamma_1\,\epsilon_0 = \eta_2\,\epsilon_0\,, \tag{8.8}$$

$$s_0 = a\,\alpha_0 + a\,\eta_2\,\left(1 - \frac{1}{b}\right)\,, \tag{8.9}$$

$$\epsilon\;\operatorname{sgn}\{\alpha_1\} = \operatorname{sgn}\left\{\frac{(\rho f)^2 + b^2 f^2 - \rho^2\,(b-1)^2}{\rho^2 + b(1+z^2)}\right\}\,, \tag{8.10}$$

$$\rho' = (b-1)\,\frac{z\,\rho(z)}{\rho^2 + b\,(1+z^2)}\,, \tag{8.11}$$

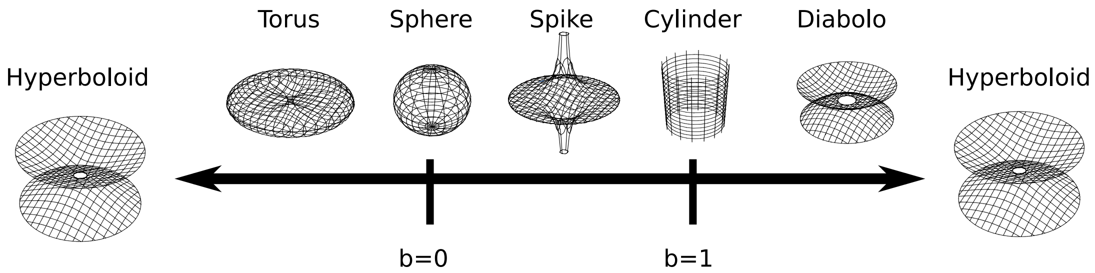

Figure 1. The solution (8.13) describes different geometries depending on the value of the parameter b.

where $b \equiv -2\,\eta_1\,\eta_2\,a$. Equation (8.11) is the BPS differential equation that determines the shape of the M2 brane. Upon imposing these conditions, the solution takes the form

$$t = \sigma^0\,, \qquad \varphi = w\sigma^1\,, \qquad z = \sigma^2\,, \tag{8.12a}$$

$$\xi^1 = \xi^2 = \xi^3 = -\xi^4 = \frac{\eta_1}{2}\,(1-b)\,\sigma^0 - \frac{1}{2}\,\eta_1\,\eta_2\,b\,\alpha_1\,\sigma^1\,. \tag{8.12b}$$

where we have made a coordinate redefinition $\alpha_0\sigma^0 + \alpha_1\sigma^1 + \alpha_2\sigma^2 \to w\sigma^1$ (w is a winding number). The general solution of equation (8.11) is given by

$$z^2 = {r_0}^2\,\rho^c - 1 - \rho^2\,, \tag{8.13}$$

where r_0 is an integration constant and

$$c = \frac{2\,b}{b-1}\,. \tag{8.14}$$

The solution is symmetric under $z \to -z$ and it has an important feature: it always crosses the $z = 0$ hyperplane smoothly. To see this, we differentiate the equation (8.13) with respect to ρ,

$$2\,z\,\frac{\mathrm{d}z}{\mathrm{d}\rho} = c\,{r_0}^2\,\rho^{c-1} - 2\,\rho\,, \tag{8.15}$$

and consider the limit $z \to 0$. From (8.13), it is easy to see that in this limit the r.h.s. of (8.15) does not vanish. Therefore, when $z \to 0$, one has $\partial_\rho z \to \infty$ (or $\rho'(0) = 0$), which is the required condition for a smooth transition.

From (8.7) and (8.8) it follows that the spinor must satisfy four independent, compatible constraints, so the solution will preserve at least $\frac{1}{16}$ of the supersymmetries. The condition (8.10) is non-trivial, but it can be shown that the solutions described below do satisfy it.

8.2 Brane scanning

The solution (8.13) describes membranes of diverse geometries depending on the value of the constant b, as shown in figure 1. Generically, these membranes have angular momenta J_φ, J_i in the φ and ξ^i directions (general formulas are given in appendix C). The standard

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expression for the Hamiltonian in the gauge $t = \sigma^0$ leads to the general formula

$$E = \frac{1}{R}\left(2|J_\varphi| + \sum_{i=1}^{4} |J_i|\right) + \frac{2T_2}{R}\int d\sigma_1 d\sigma_2\, \mathcal{L}\,. \tag{8.16}$$

As shown in [12], for these solutions the Lagrangian $\mathcal{L}$ becomes a total derivative. This implies that the last term vanishes in the case of M2 branes without boundaries. The resulting energy saturates the bound that one finds from the superalgebra, which in case of rotation in both AdS_4 and S^7 is a slight generalization of the results of section 4. Indeed, the term $2|J_\varphi|$ comes from the contribution $\tilde{\gamma}^0\tilde{\gamma}_{23}M^{23} = \tilde{\gamma}^0\tilde{\gamma}_{23}J_\varphi$. Since the matrices $\tilde{\gamma}^0\tilde{\gamma}_{23}$ and $\tilde{\gamma}^0$ commute, they can be simultaneously diagonalized and the eigenvalues of the matrix $\{Q_a, Q_b\}$ are $P^0 \pm J_\varphi \pm \hat{b}_i$, leading to the bound $P^0 \geq \frac{1}{2}\left(2|J_\varphi| + \sum_{i=1}^{4} |J_i|\right)$.

For uncompact M2 branes, the last term in (8.16) will give a non-vanishing contribution to the energy.

8.2.1 Giant spherical graviton

This appears for $b = 0$ (which implies $c = 0$). The solution (8.13) then becomes

$$z^2 + \rho^2 = {r_0}^2 - 1\,, \tag{8.17}$$

which is the equation of a sphere of radius $R_0 \equiv \sqrt{{r_0}^2 - 1}$ in cylindrical coordinates. In this case our ansatz reads

$$t = \sigma_0\,, \quad \varphi = w\sigma_1\,, \quad z = \sigma_2\,, \quad \xi^i = \frac{\eta_1}{2}\,\sigma^0\,. \tag{8.18}$$

From the formulas of appendix C one finds that this solution has $J_\varphi = 0$ and $J_i = \eta_1\, e_i\, e_4\, {\mu_i}^2\, \pi\, |w|\, T_2\, R^3\, R_0$. The energy is $E = \pi\, |w|\, T_2\, R^2\, R_0$, and therefore $E = \frac{1}{R}\sum_{i=1}^{4} |J_i|$.

8.2.2 Cylinder

For $b = 1$, the constant c tends to infinity and the solution (8.13) is no longer valid. We have to return to the original equation (8.11), which now gives $\rho' = 0$, so the radius $\rho = \rho_0$ of the cylinder is constant and arbitrary. Using the formulas of appendix C it can be easily shown that this is the only case where the angular momenta $J_\varphi,\ J_i$ vanish.

The energy is $E = \frac{1}{2}\, T_2\, \pi\, |w|\, R^2\, L$, where L regularizes the (infinite) length in z direction. Note that it is independent of ρ_0, i.e. expanding the cylinder does not cost any energy, which is consistent with the fact that solution exists for arbitrary radius. This is why the M2-brane can be in equilibrium in spite of the fact that $J_\varphi = J_i = 0$.

8.2.3 Giant spike

This appears for $0 < b < 1$. In this interval for b, the constant c covers all the negative real numbers, $c < 0$, so we can write it as $c = -|c|$, and the solution becomes

$$z^2 = \frac{r_0^2}{\rho^{|c|}} - 1 - \rho^2\,. \tag{8.19}$$

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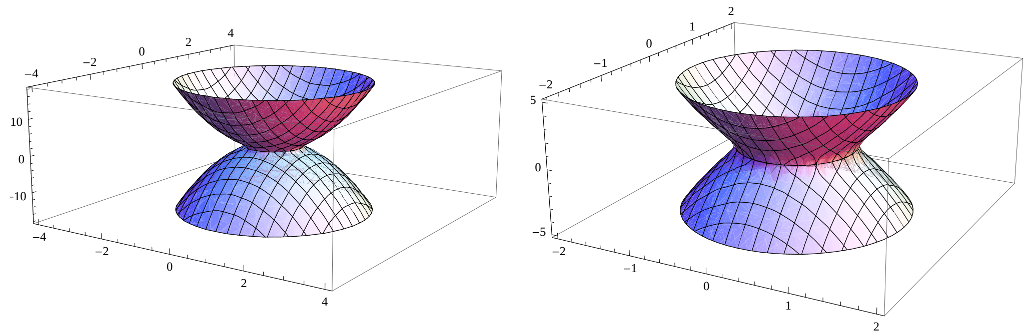

(a) General view of the giant diabolo.

(b) In this close up we can see that the transition between the two lobes is smooth.

Figure 2. The giant diabolo for $b = 1.8$ and $r_0^2 = 6$.

At $z = 0$, ρ has a unique, non-vanishing value. As shown above, the transition between $z > 0$ and $z < 0$ is smooth. At $z \to \pm\infty$, one has $\rho \to 0$, and this solution takes the form of a bulb with a spike, which in the dimensionally reduced theory can be interpreted as an open string stretched to infinity. This solution was found by Nishioka and Takayanagi in [12]. Now the energy picks a contribution from the boundary at infinity: $E = \frac{1}{R}\Big(2|J_\varphi| + \sum_{i=1}^4 |J_i|\Big) + \frac{1}{2}\, T_2\, \pi\, |w|\, R^2\, L$.

8.2.4 Hyperboloid

In the limit that b tends to $\pm$ infinity, the exponent c in (8.13) approaches the fixed value $c = 2$. The solution (8.13) now reads

$$(r_0^2 - 1)\rho^2 - z^2 = 1\,, \tag{8.20}$$

which is the equation of an hyperboloid. Note that $r_0^2 > 1$ for a real solution. It has finite $J_\varphi/(bL)$, J_i/L with $2|J_\varphi|/\sum_i |J_i| = |b| \to \infty$.

8.2.5 Giant diabolo

Consider now $b > 1$. Then the exponent c in (8.13) is always greater than two. At $z = 0$, ρ again takes a unique, finite value. At $z \to \pm\infty$, the ρ^c term dominates and

$$z \sim \pm r_0 \rho^{c/2}\,, \tag{8.21}$$

where $c/2 > 1$. This geometry resembles the shape of a diabolo, as can be seen in figure 2. A difference with the hyperboloid ($c = 2$) is that the diabolo exhibits a transition between negative and positive curvature at a certain value of ρ. We recall that the solution has angular momentum both in φ and ξ^i directions and the general formula for the energy is $E = \frac{1}{R}\Big(2|J_\varphi| + \sum_{i=1}^4 |J_i|\Big) + \frac{1}{2}\, T_2\, \pi\, |w|\, R^2\, L$.

8.2.6 Giant torus

Now we assume a negative, but finite, value of the constant $b = -|b|$. In this case,

$$c = \frac{2|b|}{|b|+1} < 2\,, \tag{8.22}$$

and solution (8.13) becomes

$$z^2 = r_0^2 \rho^c - 1 - \rho^2\,, \qquad 0 < c < 2\,. \tag{8.23}$$

Since $c < 2$, the last term, which has a negative coefficient, dominates at large ρ. Therefore z^2 can be positive only for ρ less than some maximum value ρ_M, where $z^2 = 0$. Similarly, the presence of the "-1" on the r.h.s. shows that ρ cannot be below a certain minimum value ρ_m, where $z = 0$ again. In short, when $0 < c < 2$, $z^2 > 0$ implies that ρ takes values in a finite interval $[\rho_m,\ \rho_M]$. Since z is a continuous function, it will have a maximum in this range. In conclusion, eq. (8.23) represents a torus-like geometry. This is, indeed, the giant torus configuration found in [12]. Being a compact M2 brane, one finds the simple relation $E = \frac{1}{R}\Big(2|J_\varphi| + \sum_{i=1}^4 |J_i|)\Big)$.

In conclusion, the solutions depends on two parameters, r_0, that characterizes a scale, and b. As the parameter b is varied from $-\infty$ to ∞ one witnesses different transitions of the geometry, as illustrated by figure 1.

9 Summary

Summarizing, in the first part of this paper (sections 3–7), we have investigated the following class of solutions

$$t = \omega_0 \sigma^0\,, \quad r = 0\,, \qquad Z_i = R\ \mu_i\ e^{i(\omega_i\sigma^0 + m_i\sigma^1 + n_i\sigma^2)}\,. \tag{9.1}$$

We identified two subclasses of supersymmetric solutions:

1. Supersymmetric "regular" M2 brane solutions

$$Z_1 = R\ \mu_1\ e^{i\sigma^1}, \quad Z_3 = R\ \mu_3\ e^{i\sigma^2}, \quad Z_2 = R\ \mu_2\ e^{i(\omega_2\sigma^0 + m\sigma^1)}, \quad Z_4 = R\ \mu_4\ e^{i(\omega_4\sigma^0 + n\sigma^2)}, \tag{9.2}$$

   with $m,\ n,\ \omega_2,\ \omega_4$ determined in terms of μ_i (up to signs). They are tensionless and non-collapsed; for generic values of the parameters they preserve 1/8 of the supersymmetries.

2. 1/4 and 1/8 supersymmetric collapsed M2 brane solutions

$$Z_i = R\ \mu_i\ e^{i(\omega_i\sigma^0 + m_i\sigma^1)}\,, \tag{9.3}$$

   where the amount of preserved supersymmetries depends on the values of the parameters. The different cases were analyzed in detail in section 6.3 and in appendix B. The parameters are subject to the relations (5.14) and (5.16).

As discussed, the solutions admit globally non-trivial generalizations, which can be obtained by redefinitions of σ^i.

In all cases, supersymmetry is achieved in the same limit where $E, J \to \infty$ and the M2 branes become tensionless, i.e. the determinant of the world-volume metric vanishes. This is the analog of the phenomenon found in [17] for strings. The main difference between the configurations 1 and 2 is that, in the first case, the M2-brane extends in two directions, which wrap around ξ^i coordinates, while in the second case the membrane is collapsed to a string and extends in a single direction.

Our configurations have also some similarity with the BMN configurations [5] in the sense that in both cases they correspond to null objects moving around circles of S^5 or S^7, with $E \propto J \to \infty$. It would be interesting to see if these solutions can be used to explore special sectors in ABJM theory in the same way that the BMN limit can be used to explore a sector of $\mathcal{N} = 4$ super Yang-Mills theory.

An important difference with the BMN case is that in that case the limit corresponds to a Penrose limit of the $AdS_5 \times S^5$ space, where string theory becomes solvable, allowing for an explicit comparison between field theory and string theory results. In the present case, because the M2 branes are extended, it is meaningless to ask what is the geometry seen by the generic null configurations; in particular, it cannot be obtained as a Penrose limit. In addition, in the generic case the configurations are non-perturbative from the viewpoint of string theory. Nevertheless, it is possible that a study of small fluctuations around these configurations could unveil an interesting sector of the quantum spectrum on AdS_4 and thence of ABJM $\mathcal{N} = 6$ Chern-Simons theory.

Finally, in section 8, we have revisited the supersymmetric giant graviton solutions found in [12] representing giant tori and spiky M2 branes. We re-derived the supersymmetric conditions in cylindrical coordinates, which turn out to be highly convenient to investigate the solutions in different regimes. This has unveiled a number of interesting supersymmetric uncompact M2 brane objects, including a cylinder, a hyperboloid and the giant diabolo, that extend up to the boundary of AdS_4. They should correspond to deformations ABJM by adding extra degrees of freedom (this is similar to the addition of "flavor" D7 branes to a D3 brane system).

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A Useful relations

In the main text we have defined the completely antisymmetric quantities,

$$\begin{aligned} \beta^{ij}_{\alpha\beta} &= \beta^i_\alpha \beta^j_\beta - \beta^j_\alpha \beta^i_\beta \,, \\ \beta^{ijk} &= \beta^i_0\, \beta^{jk}_{12} + \beta^j_0\, \beta^{ki}_{12} + \beta^k_0 \beta^{ij}_{12} \,. \end{aligned} \tag{A.1}$$

In terms of them we have,

$$h^{00}_c = \left(\frac{R}{2}\right)^4 \sum_{i<j} {\mu_i}^2\, {\mu_j}^2\, \left(\beta^{ij}_{12}\right)^2 \,, \tag{A.2a}$$

$$h^{11}_c = \left(\frac{R}{2}\right)^4 \left[\sum_{i<j} {\mu_i}^2\, {\mu_j}^2\, \left(\beta^{ij}_{20}\right)^2 - {\omega_0}^2\, \beta_{i,2}\, \beta^i_2\right] \,, \tag{A.2b}$$

$$h^{22}_c = \left(\frac{R}{2}\right)^4 \left[\sum_{i<j} {\mu_i}^2\, {\mu_j}^2\, \left(\beta^{ij}_{01}\right)^2 - {\omega_0}^2\, \beta_{i,1}\, \beta^i_1\right] \,, \tag{A.2c}$$

$$h^{01}_c = \left(\frac{R}{2}\right)^4 \sum_{i<j} {\mu_i}^2\, {\mu_j}^2\, \beta^{ij}_{12}\, \beta^{ij}_{20} \,, \tag{A.2d}$$

$$h^{02}_c = \left(\frac{R}{2}\right)^4 \sum_{i<j} {\mu_i}^2\, {\mu_j}^2\, \beta^{ij}_{12}\, \beta^{ij}_{01} \,, \tag{A.2e}$$

$$h^{12}_c = \left(\frac{R}{2}\right)^4 \left[\sum_{i<j} {\mu_i}^2\, {\mu_j}^2\, \beta^{ij}_{20}\, \beta^{ij}_{01} - {\omega_0}^2\, \beta_{i,1}\, \beta^i_2\right] \,. \tag{A.2f}$$

For the solution with parameters given in (3.18) we obtain the following non-zero coefficients,

$$\begin{aligned} &\beta^{12}_{01} = -4\omega_2 \,, \quad \beta^{14}_{01} = -4\omega_4 \,, \quad \beta^{24}_{01} = -4m\, \omega_4 \\ &\beta^{23}_{02} = 4\omega_2 \,, \quad \beta^{24}_{02} = 4n\, \omega_2 \,, \quad \beta^{34}_{02} = -4\omega_4 \\ &\beta^{13}_{12} = 4 \,, \quad \beta^{14}_{12} = 4n \,, \quad \beta^{23}_{12} = 4m \,, \quad \beta^{24}_{12} = 4m\, n \,. \end{aligned} \tag{A.3}$$

and

$$\beta^{123} = -8\, \omega_2 \,, \qquad \beta^{124} = -8\, n\, \omega_2 \,, \qquad \beta^{134} = 8\, \omega_4 \,, \qquad \beta^{234} = 8\, m\, \omega_4 \,. \tag{A.4}$$

These expressions are used in section 6.2.

Finally, we quote the general formula for the relation between angular momentum and energy for our family of solutions (3.24):

$$\begin{aligned} \frac{1}{RE} \sum_{i=1}^{4} |J_i| = \frac{1}{|3z-1|^{\frac{1}{2}} |z-z_0|^{\frac{1}{2}}} & \left(\frac{|z^2-z_2 z+({\mu_1}^2+{\mu_2}^2)z_0|^{\frac{1}{2}}}{|2z-{\mu_1}^2-{\mu_2}^2|^{\frac{1}{2}}} \left(\mu_1|z-{\mu_1}^2|^{\frac{1}{2}}+\mu_2|z-{\mu_2}^2|^{\frac{1}{2}}\right) \right. \\ & \left. + \frac{|z^2-z_4 z+({\mu_3}^2+{\mu_4}^2)z_0|^{\frac{1}{2}}}{|2z-{\mu_3}^2-{\mu_4}^2|^{\frac{1}{2}}} \left(\mu_3|z-{\mu_3}^2|^{\frac{1}{2}}+\mu_4|z-{\mu_4}^2|^{\frac{1}{2}}\right) \right) \end{aligned} \tag{A.5}$$

From this general expression one can see that for supersymmetric solutions with $z = 0$ the r.h.s. is equal to 1, giving rise to the BPS expression (5.10). An interesting question is if there are special values of μ_i and z for which the r.h.s. is also equal to 1, hence giving rise to the same BPS expression. This would hint on special supersymmetric configurations.

B Supersymmetry of collapsed membranes: effective string approach

As explained in section 6.3, in the case of the collapsed membrane the Γ_κ of the M2 brane is singular and cannot be used. It seems more appropriate to study the supersymmetry of this collapsed M2 brane configuration by demanding the supersymmetry condition under the "reduced" κ-symmetry matrix associated with a string-like configuration.[8] In what follows we will show that this approach correctly reproduces the number of preserved supersymmetries obtained in section 6.3 from the supersymmetry algebra. Our results will not rely on the value of the effective string tension (classically the string is tensionless, since, on-shell, the world-sheet is null).

We consider the following "reduced" κ-symmetry matrix, appropriate for string-like configurations,

$$\Gamma_\kappa = \frac{1}{\sqrt{-g}}\,\dot{X}^\mu X'^\nu \Gamma_{\mu\nu}\,, \tag{B.1}$$

where

$$\Gamma_{\mu\nu} = \frac{1}{2}\,[\Gamma_\mu,\ \Gamma_\nu]\,, \qquad \Gamma_\kappa^2 = 1\ . \tag{B.2}$$

A short computation yields,

$$\Gamma_\kappa = \frac{R^2}{4\sqrt{-g}}\left(\sum_{i<j}\mu_i\,\mu_j\,\beta^{ij}\,\gamma_{i+6}\,\gamma_{j+6} + \omega_0\sum_i \mu_i\,\beta^i_1\,\gamma_0\,\gamma_{i+6}\right)\,, \tag{B.3}$$

where $\beta^{ij} \equiv \beta^{ij}_{01} = \beta^i_0\,\beta^j_1 - \beta^j_0\,\beta^i_1$. With these ingredients, and using the relations

$$\mathcal{M}^{-1}\,\gamma_{i+6}\,\mathcal{M} = M_t^{-2}\,\Gamma_0\left(-\sum_i \mu_i\,O_i - \sum_{j\neq i}\mu_j\,\gamma_{i+6}\,\gamma_{j+6}\,O_j\,{M_i}^2\,{M_j}^2\right), \tag{B.4a}$$

$$\mathcal{M}^{-1}\,\gamma_{i+6}\,\gamma_{j+6}\,\mathcal{M} = \gamma_{i+6}\,\gamma_{j+6}\,{M_i}^2\,{M_j}^2\,, \tag{B.4b}$$

the supersymmetry condition (6.3) can be written as

$$\begin{aligned}
&\pm\left(\sum_i {\mu_i}^2{m_i}^2 - \sum_{i<j}{\mu_i}^2{\mu_j}^2\left(\frac{2\,\omega_i}{\omega_0}\,m_j - \frac{2\omega_j}{\omega_0}m_i\right)^2\right)^{1/2}\epsilon_0 \\
&= \left[-\sum_i {\mu_i}^2 m_i O_i + \sum_{i<j}\mu_i\mu_j\gamma_{i+6}\gamma_{j+6}{M_i}^2{M_j}^2\left(m_j\,O_i - m_i\,O_j - \frac{2\,\omega_i}{\omega_0}\,m_j + \frac{2\,\omega_j}{\omega_0}\,m_i\right)\right]\epsilon_0,
\end{aligned} \tag{B.5}$$

[8] We thank J. Maldacena for a discussion on this point.

where

$$M_i{}^2\, M_j{}^2 = \exp\left(-\frac{1}{2}\hat{\gamma}\gamma_0\,(\beta^i_a\,O_i + \beta^j_a\,O_j)\,\sigma^a\right)\ . \tag{B.6}$$

This equation is the starting point for analyzing the different possibilities, taking into account that the σ^a-dependence on the second term of the r.h.s. must drop out.

Let us first consider the generic case where all μ_i's are non-vanishing. We find a solution by canceling all four σ^a-dependent terms in the r.h.s of (B.5) and leaving at least two non-vanishing windings, $m_1, m_2 \neq 0$. We have

$$O_i\,\epsilon_0 = \eta_i\,\epsilon_0\,,\qquad \eta_i{}^2 = 1\,,\qquad \frac{2\,\omega^i}{\omega_0} = \eta_i - a\,m_i\,, \tag{B.7}$$

together with the constraint,

$$\sum_{i=1}^{4}\mu_i{}^2\,\eta_i\,m_i = \mp\left|\sum_{i=1}^{4}\mu_i{}^2\,\eta_i\,m_i\right|\,, \tag{B.8}$$

where the $\mp$ signs correspond to the $\pm$ signs of (B.5). Note that one possible solution of the constraint is that the sum in (B.8) vanishes. This is indeed the case as seen from the membrane equations of motion (see (5.16)), although it is not implied by the supersymmetry conditions. Due to the relation $\prod_{i=1}^{4} O_i = -1$ (which fixes $\eta_4 = -\eta_1\,\eta_2\,\eta_3$), we see that the solution preserves 1/8 of the supersymmetries. With no loss of generality we can set $\eta_{1,2,3} = 1$, $\eta_4 = -1$, as the signs of η_i can be reversed by a coordinate transformation $\xi^i \to -\xi^i$. Furthermore, the solution can be rewritten as,

$$\xi^i = \frac{1}{2}\,\omega_0\,\sigma^0 + m_i\,\sigma'^1\,,\qquad \sigma'^1 \equiv \sigma^1 - \frac{a}{2}\,\omega_0\,\sigma^0\,,\qquad i = 1,\ldots,4\,, \tag{B.9}$$

showing that the string rotates and winds with m_i in each of the four planes. This also shows that the parameter a is gauged away after the change of coordinate $\sigma \to \sigma'^1$.

Now consider the case where there is a non-trivial embedding in three planes $((12),(34),(56))$, i.e. $\mu_4 = 0$; $\mu_1{}^2 + \mu_2{}^2 + \mu_3{}^2 = 1$. A solution is obtained by cancelling three σ^a-dependent terms in the r.h.s of (B.5). One needs at least two non-vanishing m_i, i.e $m_i \neq 0$, $i = 1, 2$. Now

$$O_i\,\epsilon_0 = \eta_i\,\epsilon_0\,,\qquad \eta_i{}^2 = 1\,,\qquad \frac{2\,\omega^i}{\omega_0} = \eta_i - a\,m_i\,,\qquad i = 1,2,3\,, \tag{B.10}$$

together with the constraint (B.8). From (B.10) we see that the solution preserves 1/8 of the supersymmetries.

Finally, we consider the two-plane case $\mu_3 = \mu_4 = 0$; $\mu_1{}^2 + \mu_2{}^2 = 1$ (thus we take the (12) and (34)-planes). We demand that the σ^a-dependent term vanishes, leaving at least two non-vanishing winding numbers, $m_1, m_2 \neq 0$. We need to impose,

$$O_i\,\epsilon_0 = \eta_i\,\epsilon_0\,,\quad \eta_i{}^2 = 1\,,\qquad \frac{2\,\omega^i}{\omega_0} = \eta_i - a\,m_i\,,\qquad i = 1,2\,, \tag{B.11}$$

together with the constraint (B.8). In view of (B.11), the solution preserves 1/4 of the supersymmetries. Thus, in all cases, the number of unbroken supersymmetries obtained from the effective string approach agrees with the results derived in section 6.3 from the superalgebra.

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C Energy and angular momenta of solutions of section 8

The solution (8.12) is characterized by the energy, some winding numbers and five angular momenta, four of which associated with rotations around the ξ^i directions and another one associated with rotations around φ. These quantities can be computed directly from the Born-Infeld action by differentiating with respect to the parameter that governs translations along the corresponding directions. For this, it is convenient to introduce a parameter ω_0 in $t = \sigma^0 \to \omega_0\, \sigma^0$. Later we will set $\omega_0 = 1$ to return to our original solution.

For the ansatz (8.2), the action becomes

$$S = -\frac{1}{2} T_2 \pi R^3 \int \mathrm{d}z \Bigg\{ \Bigg[\left(1 + \rho'^2 - f'^2\right) \left(\omega_0^2 f^2 \alpha_1^2 \rho^2 - (\alpha_1 s_0 - \alpha_0 s_1)^2 \rho^2 + s_1^2 \omega_0^2 f^2\right) \Bigg]^{1/2} + \omega_0 \rho (z\rho' - \rho) \Bigg\}, \tag{C.1}$$

where we have used (8.5) and $|m_i| = 1$. The energy and the five angular momenta can then be obtained from $E = \frac{\mathrm{d}S}{\mathrm{d}\omega_0}\Big|_{\omega_0=1}$, $J_i = {\mu_i}^2 \frac{\mathrm{d}S}{\mathrm{d}s_0}$ and $J_\varphi = \frac{\mathrm{d}S}{\mathrm{d}\alpha_0}$, leading to the following expressions:

$$\begin{aligned}
E &= \frac{\pi}{2}\, |w|\; T_2\; R^2 \int dz \left\{ \frac{\sqrt{1+\rho'^2 - f'^2}\; f^2\; (\rho^2 + b^2)}{\sqrt{f^2\; (\rho^2 + b^2) - (b-1)^2\; \rho^2}} + \rho\; (z\; \rho' - \rho) \right\} \\
J_i &= -\frac{\pi}{2}\, |w|\; T_2\; R^3\; (b-1)\; {\mu_i}^2 \eta_1\; e_i\; e_4 \int dz\; \frac{\sqrt{1+\rho'^2 - f'^2}\; \rho^2}{\sqrt{f^2\; (\rho^2 + b^2) - (b-1)^2\; \rho^2}} \\
J_\varphi &= -\frac{\pi}{4}\, |w|\; T_2\; R^3\; b\; (b-1)\; \eta_2 \int dz\; \frac{\sqrt{1+\rho'^2 - f'^2}\; \rho^2}{\sqrt{f^2\; (\rho^2 + b^2) - (b-1)^2\; \rho^2}}
\end{aligned} \tag{C.2}$$

where we have substituted the parameters by their values in (8.12), i.e. $\alpha_1 = w$ and

$$\begin{aligned}
s_0 &= \frac{1}{2}\eta_1 (1-b)\,, \qquad s_1 = -\frac{1}{2}\eta_1 \eta_2\, b\, w\,, \\
s_2 &= 0\,, \qquad\qquad\quad\;\; \alpha_0 = 0\,.
\end{aligned} \tag{C.3}$$

We recall that the signs e_i, $\eta_{1,2}$ can be chosen as $e_1 = e_2 = e_3 = 1$, $e_4 = -1$, $\eta_1 = \eta_2 = 1$.

Using the specific form of the solution (8.12),

$$z^2 = r_0^2 \rho^c - 1 - \rho^2\,, \qquad f^2 = r_0^2 \rho^c\,, \qquad c \equiv \frac{2b}{b-1}\,, \tag{C.4}$$

the formulas for the angular momenta take the simple form

$$J_i = -\pi |w|\, T_2 R^3 \mu_i^2 \eta_1 e_i e_4 \int \mathrm{d}\rho\; \frac{\rho}{\sqrt{r_0^2 \rho^c - 1 - \rho^2}}\,, \tag{C.5a}$$

$$J_\varphi = \frac{\pi}{2} |w|\, T_2 R^3 b\, \eta_2 \int \mathrm{d}\rho\; \frac{\rho}{\sqrt{r_0^2 \rho^c - 1 - \rho^2}}\,. \tag{C.5b}$$

Note that the change of integration variable from z to ρ introduces a factor of 2 which accounts for both positive and negative z integration regions. Note the simple relations

$$\frac{J_i}{J_\varphi} = -2\eta_1\eta_2 e_i e_4 \mu_i^2 \frac{1}{b}\,, \qquad \sum_i |J_i| = \frac{2}{|b|}\, |J_\varphi|\,. \tag{C.6}$$

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